

GLUING IN TENSOR TRIANGULAR GEOMETRY

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ABSTRACT. We discuss gluing of objects and gluing of morphisms in tensor triangulated categories. We illustrate the results by producing, among other things, a Mayer-Vietoris exact sequence involving Picard groups.

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INTRODUCTION

Tensor Triangular Geometry is the geometry of tensor triangulated categories. Heuristically, this contains at least Algebraic Geometry and the geometry of Modular Representation Theory but it also appears in many other areas of Mathematics, as recalled in the introduction of [1].

We will denote by \mathcal{K} a triangulated category (with suspension $T : \mathcal{K} \xrightarrow{\sim} \mathcal{K}$) equipped with a tensor product, *i.e.* an exact symmetric monoidal structure $\otimes : \mathcal{K} \times \mathcal{K} \longrightarrow \mathcal{K}$, see more in Section 1. Two key examples to keep in mind appear respectively in Algebraic Geometry, as $\mathcal{K} = D^{\text{perf}}(X)$, the derived category of perfect complexes over a quasi-compact and quasi-separated scheme X (*e.g.* a noetherian scheme), and in Modular Representation Theory, as $\mathcal{K} = kG\text{-stab}$, the stable category of finite dimensional representations modulo projective ones, for G a finite group and k a field of characteristic $p > 0$, typically dividing the order of the group.

In [1], the concept of *spectrum* $\text{Spc}(\mathcal{K})$ of such categories is introduced. It is the universal topological space in which one can define *supports* $\text{supp}(a) \subset \text{Spc}(\mathcal{K})$ for objects $a \in \mathcal{K}$ in a reasonable way. In the above two examples, the spectrum $\text{Spc}(\mathcal{K})$ is respectively isomorphic to the scheme X itself and to the projective support variety $\text{Proj } H^*(G, k)$.

One fundamental construction of [1] is the *presheaf of triangulated categories*, $U \mapsto \mathcal{K}(U)$, which associates to an open $U \subset \text{Spc}(\mathcal{K})$ a tensor triangulated category $\mathcal{K}(U)$ defined as follows. Consider $Z = \text{Spc}(\mathcal{K}) \setminus U$ the closed complement of U and

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consider the thick subcategory $\mathcal{K}_Z \subset \mathcal{K}$ of those objects $a \in \mathcal{K}$ with $\text{supp}(a) \subset Z$, i.e. those objects which ought to disappear on U . Then, the category

$$\mathcal{K}(U) := \widetilde{\mathcal{K}/\mathcal{K}_Z}$$

is defined as the idempotent completion of the Verdier quotient $\mathcal{K}/\mathcal{K}_Z$. Localization $\mathcal{K} \rightarrow \mathcal{K}/\mathcal{K}_Z$ followed by idempotent completion $\mathcal{K}/\mathcal{K}_Z \hookrightarrow \mathcal{K}(U)$ yields a restriction functor $\rho_U : \mathcal{K} \rightarrow \mathcal{K}(U)$. In the scheme example, it is an important theorem of Thomason [8] that for a quasi-compact open $U \subset X$ and for $\mathcal{K} = \text{D}^{\text{perf}}(X)$, the above $\mathcal{K}(U)$ is equivalent to $\text{D}^{\text{perf}}(U)$. This is one reason for working with idempotent complete categories. Another reason is a key result of [2] which says that if \mathcal{K} is idempotent complete and if the support of an object of \mathcal{K} decomposes into two connected components then the object itself decomposes into two direct summands accordingly, see Theorem 1.8 below.

The present paper deals with the following type of questions. Suppose that $\text{Spc}(\mathcal{K})$ is covered by two quasi-compact open subsets $\text{Spc}(\mathcal{K}) = U_1 \cup U_2$ and consider the commutative diagram of restrictions:

$$(1) \quad \begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathcal{K}(U_1) =: \mathcal{K}_1 \\ \downarrow & & \downarrow \\ \mathcal{K}_2 := \mathcal{K}(U_2) & \longrightarrow & \mathcal{K}(U_1 \cap U_2) =: \mathcal{K}_{12} . \end{array}$$

Question : Is the global category \mathcal{K} obtained by “gluing” \mathcal{K}_1 and \mathcal{K}_2 over \mathcal{K}_{12} ?

This is a very natural question but it is known to be tricky, already in Algebraic Geometry. Indeed, it is easy to find non-zero morphisms $f : a \rightarrow b$ in $\mathcal{K} = \text{D}^{\text{perf}}(X)$ such that $f|_{U_1} = 0$ and $f|_{U_2} = 0$ for an open covering $X = U_1 \cup U_2$. Over $X = \mathbb{P}_k^1$ an example is the morphism $f : \mathcal{O}(2) \rightarrow T(\mathcal{O})$ which is the third one in the exact triangle corresponding to the exact sequence $\mathcal{O} \hookrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1) \twoheadrightarrow \mathcal{O}(2)$; take for U_1 and U_2 two affine subsets. (For an exact sequence of vector bundles $E' \hookrightarrow E \twoheadrightarrow E''$ over a scheme X , the corresponding morphism $f : E'' \rightarrow T(E')$ is zero in $\text{D}^{\text{perf}}(X)$ if and only if the sequence splits.) This example also shows that the phenomenon is not pathological but observable in very common situations.

Still, the problem admits a nice solution, as explained in our main results:

Theorem (Mayer-Vietoris for morphisms, see Thm. 2.11). *In the above situation (1), given two objects $a, b \in \mathcal{K}$, there exists a long exact sequence:*

$$\cdots \text{Mor}_{\mathcal{K}_{12}}(Ta, b) \xrightarrow{\partial} \text{Mor}_{\mathcal{K}}(a, b) \rightarrow \text{Mor}_{\mathcal{K}_1}(a, b) \oplus \text{Mor}_{\mathcal{K}_2}(a, b) \rightarrow \text{Mor}_{\mathcal{K}_{12}}(a, b) \xrightarrow{\partial} \cdots$$

The connecting homomorphism $\partial : \text{Mor}_{\mathcal{K}_{12}}(Ta, b) \rightarrow \text{Mor}_{\mathcal{K}}(a, b)$ is defined in Construction 2.9. The other ones are the obvious restrictions and differences thereof.

Theorem (Gluing of two objects, see Thm. 3.3). *In the above situation (1), given two objects $a_1 \in \mathcal{K}(U_1)$ and $a_2 \in \mathcal{K}(U_2)$ and an isomorphism $\sigma : a_1 \xrightarrow{\sim} a_2$ in $\mathcal{K}(U_1 \cap U_2)$, there exists an object $a \in \mathcal{K}$ which becomes isomorphic to a_i in $\mathcal{K}(U_i)$ for $i = 1, 2$. Moreover, this gluing is unique up to (non-unique) isomorphism.*

We can extend the above result to three open subsets and three objects, at the cost of possibly losing unicity of the gluing:

Corollary (Gluing of three objects, see Cor. 3.4). *Let $\mathrm{Spc}(\mathcal{K}) = U_1 \cup U_2 \cup U_3$ be a covering by quasi-compact open subsets. Consider three objects $a_i \in \mathcal{K}(U_i)$ for $i = 1, 2, 3$ and three isomorphisms $\sigma_{ij} : a_j \xrightarrow{\sim} a_i$ in $\mathcal{K}(U_i \cap U_j)$ for $1 \leq i < j \leq 3$. Suppose that the cocycle relation $\sigma_{12} \circ \sigma_{23} = \sigma_{13}$ is satisfied in $\mathcal{K}(U_1 \cap U_2 \cap U_3)$. Then there exists an object $a \in \mathcal{K}$, isomorphic to a_i in $\mathcal{K}(U_i)$ for $i = 1, 2, 3$.*

In general, we do not know if this gluing is possible with more than three open subsets. Nevertheless, in Theorem 3.6, we give elementary conditions under which the gluing is possible for arbitrary coverings.

Then, we apply the main results to obtain an exact sequence involving Picard groups. For us, the Picard group, $\mathrm{Pic}(\mathcal{K})$, is the set of isomorphism classes of invertible objects in \mathcal{K} , with the tensor product as multiplication. In Algebraic Geometry, $\mathrm{Pic}(\mathrm{D}^{\mathrm{perf}}(X))$ is the usual Picard group of X up to possible shifts, see Prop. 4.4. On the other hand, $\mathrm{Pic}(kG\text{-stab})$ is nothing but the group of endo-trivial representations, which is one of the fundamental invariants of Modular Representation Theory. In the next statement, we denote by $\mathbb{G}_m(\mathcal{K}) = \mathrm{Mor}_{\mathcal{K}}(1, 1)^{\times}$ the abelian group of automorphisms of the \otimes -unit object $1 \in \mathcal{K}$.

Theorem (Mayer-Vietoris for Picard groups, see Thm. 4.6). *Let $\mathrm{Spc}(\mathcal{K}) = U_1 \cup U_2$ with U_i quasi-compact. Then there is half a long exact sequence :*

$$\begin{aligned} & \cdots \longrightarrow \mathrm{Mor}_{\mathcal{K}(U_1 \cap U_2)}(T1, 1) \xrightarrow{1+\partial} \\ & \xrightarrow{1+\partial} \mathbb{G}_m(\mathcal{K}) \longrightarrow \mathbb{G}_m(\mathcal{K}(U_1)) \oplus \mathbb{G}_m(\mathcal{K}(U_2)) \longrightarrow \mathbb{G}_m(\mathcal{K}(U_1 \cap U_2)) \xrightarrow{\delta} \\ & \xrightarrow{\delta} \mathrm{Pic}(\mathcal{K}) \longrightarrow \mathrm{Pic}(\mathcal{K}(U_1)) \oplus \mathrm{Pic}(\mathcal{K}(U_2)) \longrightarrow \mathrm{Pic}(\mathcal{K}(U_1 \cap U_2)). \end{aligned}$$

To the left, we have the Mayer-Vietoris long exact sequence, the homomorphism ∂ is as before and the non-labelled morphisms are again the obvious restrictions and (multiplicative) differences of restrictions. The new homomorphism

$$\delta : \mathbb{G}_m(\mathcal{K}(U_1 \cap U_2)) \rightarrow \mathrm{Pic}(\mathcal{K})$$

assigns to a unit $\sigma \in \mathbb{G}_m(\mathcal{K}(U_1 \cap U_2))$ the invertible object obtained by gluing two copies of the objects $1 \in \mathcal{K}(U_1)$ and $1 \in \mathcal{K}(U_2)$ along $\sigma : 1 \xrightarrow{\sim} 1$ on $U_1 \cap U_2$.

It would be very interesting to continue this sequence to the right, say, with Brauer groups of Azumaya algebras. Although this is still work in progress, the authors do not know yet whether such an extension is possible in general. Neither do we know what the Brauer group of $\mathcal{K} = kG\text{-stab}$ should be, for instance.

In fact, in Modular Representation Theory, applying the above results to $\mathcal{K} = kG\text{-stab}$ gives us a way to construct endo-trivial kG -modules from any Čech \mathbb{G}_m -cocycle over the projective support variety $\mathrm{Proj} H^{\bullet}(G, k)$, as long as the involved covering has at most three open pieces. In particular, the map δ of the last result might be of interest to representation theorists and we do not know if it has been studied, even in special cases. Dave Benson and Jon Carlson suggested a possible link with the recent article [4]. This will be investigated in future work.

Using the conditional gluing of more than three objects, we obtain the following result (Thm 4.7), which relates invertible modules over the spectrum $\mathrm{Spc}(\mathcal{K})$ and invertible objects in \mathcal{K} . See more comments in Remark 4.8.

Theorem. *Suppose that $\mathrm{Mor}_{\mathcal{K}(U)}(T1, 1) = 0$ for every quasi-compact open subsets $U \subset \mathrm{Spc}(\mathcal{K})$. Then, gluing induces an injective homomorphism from the first Čech cohomology of $\mathrm{Spc}(\mathcal{K})$ with coefficients in \mathbb{G}_m into the Picard group of \mathcal{K}*

$$\check{H}^1(\mathrm{Spc}(\mathcal{K}), \mathbb{G}_m) \hookrightarrow \mathrm{Pic}(\mathcal{K}).$$

We end the paper with the following formulation of Mayer-Vietoris:

Theorem (Excision, see Thm. 5.1). *Let $Y \subset U \subset \mathrm{Spc}(\mathcal{K})$. Assume that Y is closed with quasi-compact complement and that U is open and quasi-compact. Then the restriction functor $\mathcal{K} \rightarrow \mathcal{K}(U)$ induces an equivalence between the subcategories of objects supported on Y , that is, $\mathcal{K}_Y \xrightarrow{\sim} \mathcal{K}(U)_Y$.*

1. BASICS ABOUT TENSOR TRIANGULATED CATEGORIES AND THEIR GEOMETRY

We survey the main concepts and results of [1] and [2]. Standard notions about triangulated categories can be found in Verdier [9] or Neeman [6].

Definitions 1.1. A *tensor triangulated category* $(\mathcal{K}, \otimes, 1)$ is an essentially small triangulated category \mathcal{K} with a symmetric monoidal structure $\otimes : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$, $(a, b) \mapsto a \otimes b$. We have in particular $a \otimes b \cong b \otimes a$ and $1 \otimes a \cong a$ for the *unit* $1 \in \mathcal{K}$. We assume moreover that the functors $a \otimes -$ and $- \otimes b$ are exact for every $a, b \in \mathcal{K}$ and that the usual diagram

$$\begin{array}{ccc} T(a) \otimes T(b) & \xrightarrow{\cong} & T(T(a) \otimes b) \\ \cong \downarrow & & \downarrow \cong \\ T(a \otimes T(b)) & \xrightarrow{\cong} & T^2(a \otimes b) \end{array}$$

anticommutes. We use $T : \mathcal{K} \xrightarrow{\sim} \mathcal{K}$ to denote the translation (suspension).

A *prime ideal* $\mathcal{P} \subsetneq \mathcal{K}$ is a proper subcategory such that (1)-(4) below hold true:

- (1) \mathcal{P} is a full triangulated subcategory, i.e. $0 \in \mathcal{P}$, $T(\mathcal{P}) = \mathcal{P}$ and if $a, b \in \mathcal{P}$ and if $a \rightarrow b \rightarrow c \rightarrow T(a)$ is a distinguished triangle in \mathcal{K} then $c \in \mathcal{P}$;
- (2) \mathcal{P} is *thick*, i.e. if $a \oplus b \in \mathcal{P}$ then $a, b \in \mathcal{P}$;
- (3) \mathcal{P} is a \otimes -ideal, i.e. if $a \in \mathcal{P}$ then $a \otimes b \in \mathcal{P}$ for all $b \in \mathcal{K}$;
- (4) \mathcal{P} is *prime*, i.e. if $a \otimes b \in \mathcal{P}$ then $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

A subcategory $\mathcal{J} \subset \mathcal{K}$ satisfying (1), (2) and (3) is a *thick \otimes -ideal*.

The *spectrum* $\mathrm{Spc}(\mathcal{K})$ is the set of primes $\mathcal{P} \subset \mathcal{K}$. The *support* of an object $a \in \mathcal{K}$ is defined as the subset $\mathrm{supp}(a) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid a \notin \mathcal{P}\} \subset \mathrm{Spc}(\mathcal{K})$. The complements $U(a) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid a \in \mathcal{P}\}$ of these supports form a basis $\{U(a)\}_{a \in \mathcal{K}}$ of the so-called *Zariski topology* on the spectrum.

Theorem 1.2 ([1] Thm. 3.2). *Let \mathcal{K} be a tensor triangulated category. We have*

- (i) $\mathrm{supp}(0) = \emptyset$ and $\mathrm{supp}(1) = \mathrm{Spc}(\mathcal{K})$.
- (ii) $\mathrm{supp}(a \oplus b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$.
- (iii) $\mathrm{supp}(Ta) = \mathrm{supp}(a)$.
- (iv) $\mathrm{supp}(a) \subset \mathrm{supp}(b) \cup \mathrm{supp}(c)$ for any distinguished $a \rightarrow b \rightarrow c \rightarrow T(a)$.
- (v) $\mathrm{supp}(a \otimes b) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$.

Moreover, $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ is universal for these properties.

Notation 1.3. Let $Y \subset \mathrm{Spc}(\mathcal{K})$. We denote by \mathcal{K}_Y the full subcategory $\mathcal{K}_Y := \{a \in \mathcal{K} \mid \mathrm{supp}(a) \subset Y\}$ of those objects *supported on* Y .

Definition 1.4. We call a tensor triangulated category $(\mathcal{K}, \otimes, 1)$ *strongly closed* if there exists a bi-exact functor $\underline{\mathrm{hom}} : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \longrightarrow \mathcal{K}$ with natural isomorphisms

$$(2) \quad \mathrm{Mor}_{\mathcal{K}}(a \otimes b, c) \cong \mathrm{Mor}_{\mathcal{K}}(a, \underline{\mathrm{hom}}(b, c))$$

and such that all objects are *strongly dualizable*, i.e. the natural morphism

$$(3) \quad D(a) \otimes b \xrightarrow{\sim} \underline{\mathrm{hom}}(a, b)$$

is an isomorphism for all $a, b \in \mathcal{K}$, where we denote by $D(a)$ the dual $D(a) := \underline{\mathrm{hom}}(a, 1)$ of an object $a \in \mathcal{K}$. More details can be found in [5, App. A], for instance. It follows from (3) that $D^2(a) \cong a$ for all $a \in \mathcal{K}$; see for instance [5, Thm. A.2.5 (b)].

Proposition 1.5 ([2] Cor. 2.5). *Let \mathcal{K} be a strongly closed tensor triangulated category and let $a \in \mathcal{K}$ be an object. Then $\mathrm{supp}(a) = \emptyset$ if and only if $a = 0$.*

Proposition 1.6 ([2] Cor. 2.8). *Let \mathcal{K} be a strongly closed tensor triangulated category. Suppose that the supports of two objects do not meet: $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset$. Then there is no non-trivial morphism between them: $\mathrm{Mor}_{\mathcal{K}}(a, b) = 0$.*

Proposition 1.7. *Let \mathcal{K} be a strongly closed tensor triangulated category. A morphism $f : a \rightarrow b$ in \mathcal{K} is an isomorphism if and only if it is an isomorphism in \mathcal{K}/\mathcal{P} for all $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$.*

Proof. This easily follows from the fact that a morphism f in a triangulated category is an isomorphism if and only if $\mathrm{cone}(f) = 0$. Thus if f is an isomorphism in \mathcal{K}/\mathcal{P} we have that $\mathrm{cone}(f) \in \mathcal{P}$. If this is true for all $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ we have that $\mathrm{supp}(\mathrm{cone}(f)) = \emptyset$ which implies that $\mathrm{cone}(f) = 0$ by Proposition 1.5. \square

Theorem 1.8 ([2] Thm. 2.11). *Let \mathcal{K} be a strongly closed tensor triangulated category. Assume that \mathcal{K} is idempotent complete. Then, if the support of an object $a \in \mathcal{K}$ can be decomposed as $\mathrm{supp}(a) = Y_1 \cup Y_2$ for disjoint closed subsets $Y_1, Y_2 \subset \mathrm{Spc}(\mathcal{K})$, with each open complement $\mathrm{Spc}(\mathcal{K}) \setminus Y_i$ quasi-compact, then the object itself can be decomposed as a direct sum $a \simeq a_1 \oplus a_2$ with $\mathrm{supp}(a_i) = Y_i$ for $i = 1, 2$.*

Remark 1.9. Recall that an additive category \mathcal{K} is *idempotent complete* (or pseudo-abelian or karoubian) if all idempotents of all objects split, that is, if $e \in \mathrm{Mor}_{\mathcal{K}}(a, a)$ with $e^2 = e$ then the object a decomposes as a direct sum $a \simeq a' \oplus a''$ on which e becomes $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, that is, $a \simeq \mathrm{Im}(e) \oplus \mathrm{Ker}(e)$. One can always “idempotent complete” an additive category $\mathcal{K} \hookrightarrow \tilde{\mathcal{K}}$ and $\tilde{\mathcal{K}}$ inherits a unique triangulation from \mathcal{K} , see [3].

Definitions 1.10. Let \mathcal{K} be an idempotent complete strongly closed tensor triangulated category.

Let U be a quasi-compact open subset of $\mathrm{Spc}(\mathcal{K})$, and let us denote by $Z = \mathrm{Spc}(\mathcal{K}) \setminus U$ its closed complement. We denote by $\mathcal{L}(U) = \mathcal{K}/\mathcal{K}_Z$ the Verdier localization with respect to \mathcal{K}_Z (which can be realized by keeping the same objects as \mathcal{K} and by inverting all morphisms whose cone belongs to \mathcal{K}_Z , by means of calculus of fractions). We denote by $\widehat{\mathcal{K}(U)} = \widehat{\mathcal{L}(U)}$ its idempotent completion. We have a fully faithful cofinal morphism $\mathcal{L}(U) \hookrightarrow \widehat{\mathcal{K}(U)}$ (*cofinal* is sometimes called *dense*, like in [7], and means that every object of the big category is a direct summand of an object of the small one).

For $U = \mathrm{Spc}(\mathcal{K})$, by Proposition 1.5, we have $\mathcal{L}(U) = \mathcal{K} = \widetilde{\mathcal{K}} = \mathcal{K}(U)$ since we assume \mathcal{K} idempotent complete. If $U \subset V$ we denote by $\rho_{V,U} : \mathcal{L}(V) \rightarrow \mathcal{L}(U)$ the localization functor and we also denote by

$$\rho_{V,U} : \mathcal{K}(V) \rightarrow \mathcal{K}(U)$$

the induced functor. When $V = \mathrm{Spc}(\mathcal{K})$, we simply write $\rho_U : \mathcal{K} \rightarrow \mathcal{K}(U)$ for $\rho_{V,U}$.

For two objects a, b of \mathcal{K} we denote by

$$\mathrm{Mor}_U(a, b) := \mathrm{Mor}_{\mathcal{L}(U)}(\rho_U(a), \rho_U(b)) = \mathrm{Mor}_{\mathcal{K}(U)}(\rho_U(a), \rho_U(b))$$

the set of morphisms between $\rho_U(a)$ and $\rho_U(b)$ in $\mathcal{L}(U)$ or equivalently in its idempotent completion $\mathcal{K}(U)$; for simplicity, we might speak of “morphisms between a and b in $\mathcal{K}(U)$ ”, or simply of “morphisms between a and b over U ”.

Proposition 1.11. *For $U \subset \mathrm{Spc}(\mathcal{K})$ quasi-compact and open, the restriction functor $\rho_U : \mathcal{K} \rightarrow \mathcal{K}(U)$ induces a homeomorphism $\mathrm{Spc}(\mathcal{K}(U)) \xrightarrow{\sim} U$, under which $\mathrm{supp}(\rho_U(a)) = U \cap \mathrm{supp}(a)$, for any object $a \in \mathcal{K}$.*

Proof. In fact, by [1] Cor. 3.14, $\mathrm{Spc}(\mathcal{K}(U)) = \mathrm{Spc}(\mathcal{K}/\mathcal{K}_Z)$ and by *loc. cit.* Prop. 3.11, the localization functor induces a homeomorphism between $\mathrm{Spc}(\mathcal{K}/\mathcal{K}_Z)$ and the subspace $V := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{K}_Z \subset \mathcal{P}\}$ of $\mathrm{Spc}(\mathcal{K})$. So, it suffices to check that $V = U$. The last equality $\mathrm{supp}(\rho_U(a)) = U \cap \mathrm{supp}(a)$ will then be a general fact about the functor $\mathrm{Spc}(-)$, see *loc. cit.* Prop. 3.6.

Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$. By the classification of thick \otimes -ideals, *loc. cit.* Thm. 4.10, we have $\mathcal{P} \in V$, i.e. $\mathcal{K}_Z \subset \mathcal{P}$, if and only if $Z = \mathrm{supp}(\mathcal{K}_Z) \subset \mathrm{supp}(\mathcal{P}) \stackrel{\mathrm{def.}}{=} \bigcup_{a \in \mathcal{P}} \mathrm{supp}(a)$. By taking complements, this is equivalent to $\bigcap_{a \in \mathcal{P}} U(a) \subset U$, where $U(a) = \mathrm{Spc}(\mathcal{K}) \setminus \mathrm{supp}(a) = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}) \mid a \in \mathcal{Q}\}$. Tautologically, $\bigcap_{a \in \mathcal{P}} U(a) = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{P} \subset \mathcal{Q}\}$. The latter set is contained in U if and only if $\mathcal{P} \in U$: one direction is trivial and the other one uses that Z is specialization closed, see *loc. cit.* Prop. 2.9. So, $\mathcal{P} \in V$ if and only if $\mathcal{P} \in U$, as was left to check. \square

Remark 1.12. The above result cannot hold without assuming U quasi-compact since $\mathrm{Spc}(\mathcal{K})$ is quasi-compact for any \mathcal{K} . It is used above to insure $Z = \mathrm{supp}(\mathcal{K}_Z)$.

* * *

We end this Section with some general facts about triangulated categories.

Lemma 1.13. *Let \mathcal{K} be a triangulated category. Then for every distinguished triangle in which one object decomposes into two direct summands*

$$a \xrightarrow{s} b \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} c_1 \oplus c_2 \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} Ta$$

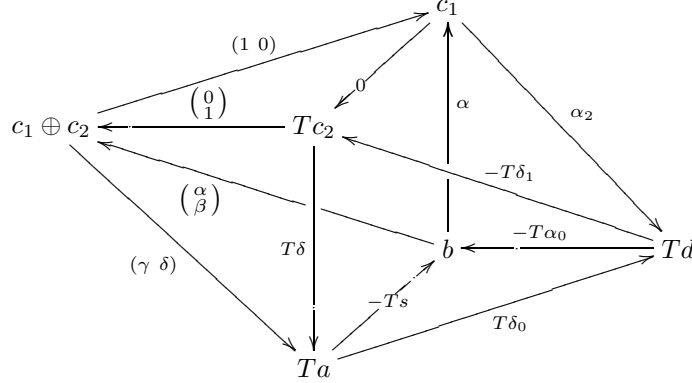
there exist two objects, d and e , and four distinguished triangles:

$$\begin{array}{ccc} d \xrightarrow{\alpha_0} b \xrightarrow{\alpha} c_1 \xrightarrow{\alpha_2} Td & a \xrightarrow{\delta_0} d \xrightarrow{\delta_1} c_2 \xrightarrow{\delta} Ta \\ e \xrightarrow{\beta_0} b \xrightarrow{\beta} c_2 \xrightarrow{\beta_2} Te & a \xrightarrow{\gamma_0} e \xrightarrow{\gamma_1} c_1 \xrightarrow{\gamma} Ta \end{array}$$

where $\alpha_2 = T\delta_0\gamma$, $\delta_1 = \beta\alpha_0$, $\beta_2 = T\gamma_0\delta$ and $\gamma_1 = \alpha\beta_0$. Moreover, the morphism s factors as $s = \alpha_0\delta_0 = \beta_0\gamma_0$.

In particular, we have $\mathrm{cone}(\alpha) \simeq \mathrm{cone}(\delta)$ and $\mathrm{cone}(\beta) \simeq \mathrm{cone}(\gamma)$.

Proof. We will prove the existence of the first two triangles, the other two are obtained symmetrically ($c_1 \oplus c_2 \simeq c_2 \oplus c_1$). The triangles are obtained by applying the Octahedron Axiom to the equality $(1 \ 0) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha$ as displayed below:



□

Definition 1.14. We say that a commutative square as follows is a *weak push-out*

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ g \downarrow & & \downarrow h \\ c & \xrightarrow{k} & d \end{array}$$

if there exists a distinguished triangle $a \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} b \oplus c \xrightarrow{\begin{pmatrix} -h & k \end{pmatrix}} d \xrightarrow{l} Ta$ for some morphism $l : d \rightarrow T(a)$. This is justified since (d, h, k) satisfies the universal property of the push-out of f and g but without unicity of the factorization. Since such a square is then also a *weak pull-back*, we call it *weakly bicartesian*.

2. MAYER-VIETORIS FOR MORPHISMS

Definition 2.1. Let \mathcal{K} be an idempotent complete strongly closed tensor triangulated category. We say that we are in a *Mayer-Vietoris situation* if the spectrum of \mathcal{K} is covered by two quasi-compact open subsets $\mathrm{Spc}(\mathcal{K}) = U_1 \cup U_2$.

We shall denote by $Z_i = \mathrm{Spc}(\mathcal{K}) \setminus U_i$ the closed complements for $i = 1, 2$. Recall the important Definitions 1.10. We will use the simplified notation $\rho_i = \rho_{U_i}$ and

$\rho_{ij} = \rho_{U_i, U_i \cap U_j}$ for the restriction functors. We have the commutative diagram :

$$\begin{array}{ccccc}
 \mathcal{K} & \xrightarrow{\rho_1} & \mathcal{K}(U_1) & & \\
 \downarrow \rho_2 & \searrow \rho_1 & \downarrow \rho_{12} & \nearrow & \\
 & \mathcal{L}(U_1) & & & \\
 & \downarrow \rho_{12} & & & \\
 & \mathcal{L}(U_2) & \xrightarrow{\rho_{21}} & \mathcal{L}(U_1 \cap U_2) & \\
 & \nearrow & & \searrow & \\
 \mathcal{K}(U_2) & \xrightarrow{\rho_{21}} & \mathcal{K}(U_1 \cap U_2) & &
 \end{array}$$

where \rightarrow denotes a Verdier localization and \hookrightarrow a fully faithful cofinal embedding.

Remark 2.2. We do not really need to have U_1 and U_2 open and it would be enough to assume that they are arbitrary intersections of quasi-compact open subsets. Indeed, the key result taken from [2], Theorem 1.8, holds in this generality. Therefore, everything below holds in similar generality. We stick to open pieces because this is closer to the reader's understanding of a Mayer-Vietoris framework.

Definition 2.3. Let $U \subset \mathrm{Spc}(\mathcal{K})$ be a (quasi-compact) open with closed complement Z . A morphism $s : a \rightarrow b$ in \mathcal{K} is called a *U -isomorphism* if it is an isomorphism in $\mathcal{L}(U)$, or equivalently in $\mathcal{K}(U)$. This is also equivalent to saying that $\mathrm{cone}(s)$ belongs to \mathcal{K}_Z which also reads $\mathrm{supp}(\mathrm{cone}(s)) \cap U = \emptyset$.

Lemma 2.4. Consider a weakly bicartesian square in \mathcal{K} (Def. 1.14) :

$$\begin{array}{ccc}
 a & \xrightarrow{f} & b \\
 g \downarrow & & \downarrow h \\
 c & \xrightarrow{k} & d
 \end{array}$$

Let $U \subset \mathrm{Spc}(\mathcal{K})$ be open. Then f is a U -isomorphism if and only if k is.

Proof. There exists a distinguished triangle $T^{-1}d \longrightarrow a \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} b \oplus c \xrightarrow{\begin{pmatrix} -h & k \end{pmatrix}} d$. By Lemma 1.13, $\mathrm{cone}(f) \simeq \mathrm{cone}(k)$ and the result follows. \square

Remark 2.5. In a Mayer-Vietoris situation, a morphism which is both a U_1 - and a U_2 -isomorphism must be an isomorphism since the support of its cone is empty.

Lemma 2.6. In a Mayer-Vietoris situation, suppose that $s : a \rightarrow b$ is a $U_1 \cap U_2$ -isomorphism. Then s may be factored as $s = s_1 s_2$ where s_i is a U_i -isomorphism.

Proof. By hypothesis we have that $\mathrm{cone}(s) \in \mathcal{K}_{Z_1 \cup Z_2}$. Thus by Theorem 1.8 $\mathrm{cone}(s)$ may be written as $\mathrm{cone}(s) \simeq c_1 \oplus c_2$ where $c_i \in \mathcal{K}_{Z_i}$. Now use Lemma 1.13 which tells that $s = \alpha_0 \delta_0$ and that $\mathrm{cone}(\alpha_0) \simeq c_1$ and $\mathrm{cone}(\delta_0) \simeq c_2$. \square

Remark 2.7. One can actually prove that the above factorization is essentially unique but we shall not use this fact below.

Lemma 2.8. *In a Mayer-Vietoris situation, consider a commutative diagram :*

$$\begin{array}{ccc} a & \xrightarrow{s_1} & b \\ t_2 \downarrow & & \downarrow s_2 \\ c & \xrightarrow{t_1} & d. \end{array}$$

Assume that s_i and t_i are U_i -isomorphisms for $i = 1, 2$. Then the square is weakly bicartesian.

Proof. Consider the weak push-out (e, u_1, u_2) of s_1 and t_2 and the morphism $v : e \rightarrow d$ induced by s_2 and t_1 :

$$\begin{array}{ccccc} a & \xrightarrow{s_1} & b & & \\ t_2 \downarrow & & \downarrow u_2 & \searrow s_2 & \\ c & \xrightarrow{u_1} & e & \xrightarrow{v} & d. \\ & \searrow t_1 & & & \end{array}$$

By Lemma 2.4, u_i is a U_i -isomorphism for $i = 1, 2$. By 2-out-of-3, v is both a U_1 - and a U_2 -isomorphism, hence an isomorphism (see Rem. 2.5). \square

Construction 2.9. Consider a Mayer-Vietoris situation (Def. 2.1) and two objects $a, b \in \mathcal{K}$. We define a homomorphism

$$\begin{aligned} \partial : \text{Mor}_{U_1 \cap U_2}(a, b) &\longrightarrow \text{Mor}_{\mathcal{K}}(a, T(b)) \\ g &\longmapsto \partial(g) \end{aligned}$$

as follows. Let $f s^{-1} = (a \xleftarrow{s} x \xrightarrow{f} b)$ be a fraction representing a morphism $g \in \text{Mor}_{U_1 \cap U_2}(a, b)$. The cone of the $U_1 \cap U_2$ -isomorphism s belongs to $\mathcal{K}_{Z_1 \cup Z_2} = \mathcal{K}_{Z_1} \oplus \mathcal{K}_{Z_2}$, see Theorem 1.8. So, we can chose a distinguished triangle

$$x \xrightarrow{s} a \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} c_1 \oplus c_2 \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} Tx$$

where $c_i \in \mathcal{K}_{Z_i}$ for $i = 1, 2$. Note that $\gamma\alpha + \delta\beta = 0$. Define now

$$\partial(g) := T(f) \circ \gamma \circ \alpha = -T(f) \circ \delta \circ \beta.$$

This is a morphism in $\text{Mor}_{\mathcal{K}}(a, b)$, independant of the choices, see Theorem 2.11.

Remark 2.10. Since T is an equivalence, the above construction also induces :

$$\text{Mor}_{U_1 \cap U_2}(T(a), b) \xrightarrow{\partial} \text{Mor}_{\mathcal{K}}(T(a), T(b)) \xrightarrow[\cong]{T^{-1}} \text{Mor}_{\mathcal{K}}(a, b)$$

and we also denote this homomorphism by ∂ , since no confusion should follow.

Explicitly, for a morphism $g = (T(a) \xleftarrow{s} x \xrightarrow{f} b)$ in $\text{Mor}_{U_1 \cap U_2}(T(a), b)$, choose

any distinguished triangle $a \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} c_1 \oplus c_2 \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} x \xrightarrow{s} T(a)$ with $c_i \in \mathcal{K}_{Z_i}$ for $i = 1, 2$; then we have $\partial(g) = f \circ \gamma \circ \alpha = -f \circ \delta \circ \beta \in \text{Mor}_{\mathcal{K}}(a, b)$.

Theorem 2.11 (Mayer-Vietoris for morphisms). *In a Mayer-Vietoris situation (Def. 2.1), for each pair of objects $a, b \in \mathcal{K}$, the above connecting homomorphism ∂ is well-defined and there is a long exact sequence of abelian groups*

$$\begin{aligned} & \cdots \longrightarrow \mathrm{Mor}_{U_1 \cap U_2}(Ta, b) \xrightarrow{\partial} \\ & \xrightarrow{\partial} \mathrm{Mor}_{\mathcal{K}}(a, b) \longrightarrow \mathrm{Mor}_{U_1}(a, b) \oplus \mathrm{Mor}_{U_2}(a, b) \longrightarrow \mathrm{Mor}_{U_1 \cap U_2}(a, b) \xrightarrow{\partial} \\ & \xrightarrow{\partial} \mathrm{Mor}_{\mathcal{K}}(a, Tb) \longrightarrow \cdots \end{aligned}$$

where the non-labelled homomorphisms are the obvious restrictions and differences of restrictions.

Proof. First, we have to check that the definition of $\partial(\frac{f}{s})$ given in 2.9 does not depend on the choice of the objects $c_i \in \mathcal{K}_{Z_i}$ such that $\mathrm{cone}(s) \simeq c_1 \oplus c_2$. This is easy, for other c_i must be isomorphic to the chosen ones: $c_1 \oplus c_2 \simeq c'_1 \oplus c'_2$ and $\mathrm{supp}(c_i) \subset Z_i$ forces $c_i \simeq c'_i$ for $i = 1, 2$, by Prop. 1.6. The isomorphism $c_1 \simeq c'_1$ disappears in the composition $\gamma \circ \alpha$ and a fortiori in $\partial(\frac{f}{s})$.

Then, we have to check that $\partial(\frac{f}{s})$ only depends on the class of the fraction $\frac{f}{s}$.

To see this, let $t : y \rightarrow x$ be a $U_1 \cap U_2$ -isomorphism and let $\frac{ft}{st} = (a \xleftarrow{st} y \xrightarrow{ft} b)$ be the amplified fraction. We have to show that $\partial(\frac{ft}{st}) = \partial(\frac{f}{s})$. The morphism st fits in a distinguished triangle.

$$y \xrightarrow{st} a \xrightarrow{\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}} d_1 \oplus d_2 \xrightarrow{(\gamma' \delta')} Ty$$

where $d_i \in \mathcal{K}_{Z_i}$. Comparing the triangles for s and for st yields the diagram

$$\begin{array}{ccccccc} y & \xrightarrow{st} & a & \xrightarrow{\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}} & d_1 \oplus d_2 & \xrightarrow{(\gamma' \delta')} & Ty \\ \downarrow t & & \parallel & & \downarrow \exists \epsilon & & \downarrow Tt \\ x & \xrightarrow{s} & a & \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} & c_1 \oplus c_2 & \xrightarrow{(\gamma \delta)} & Tx \end{array}$$

for some morphism ϵ . But since $\mathrm{supp}(d_i) \cap \mathrm{supp}(c_j) = \emptyset$ for $i \neq j$, by Proposition 1.6, we have that $\epsilon = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}$. Now compute

$$\partial\left(\frac{ft}{st}\right) = T(ft)\gamma'\alpha' = TfTt\gamma'\alpha' = Tf\gamma\epsilon_1\alpha' = Tf\gamma\alpha = \partial\left(\frac{f}{s}\right).$$

This proves that ∂ is well-defined. The fact that ∂ does not depend on the amplification of the fraction shows also that in order to prove that ∂ is a group homomorphism it suffices to see that $\partial(\frac{f+g}{s}) = \partial(\frac{f}{s}) + \partial(\frac{g}{s})$, which is immediate.

We now prove that the sequence is exact. It is easy to see that all consecutive compositions are zero. (Recall the notation for the restriction functors ρ_i and ρ_{ij} from Definition 2.1.) For instance, $\rho_i(\partial(\frac{f}{s})) = 0$ because $\partial(\frac{f}{s})$ factors via $c_i \in \mathcal{K}_{Z_i}$ which becomes zero over U_i . To see that $\partial \circ \rho_{ij} = 0$, we check that $\partial(\frac{f}{s}) = 0$ if s is a U_1 -isomorphism for instance. But in this case, $c_2 = 0$ and $\partial(\frac{f}{s})$ factors via c_2 .

Exactness at $\text{Mor}_{U_1}(a, b) \oplus \text{Mor}_{U_2}(a, b)$: Let $(f_1, f_2) \in \text{Mor}_{U_1}(a, b) \oplus \text{Mor}_{U_2}(a, b)$ such that $\rho_{12}(f_1) = \rho_{21}(f_2)$. Write $f_i = (a \xleftarrow{s_i} x_i \xrightarrow{g_i} b)$. Then there exist an object x and $U_1 \cap U_2$ -isomorphisms $t_i : x \rightarrow x_i$ such that the diagram

$$(4) \quad \begin{array}{ccccc} & & x_1 & & \\ & s_1 \swarrow & \uparrow t_2 & \searrow g_1 & \\ a & & x & & b \\ & s_2 \swarrow & \downarrow t_1 & \searrow g_2 & \\ & & x_2 & & \end{array}$$

is commutative. By Lemma 2.6 we know that every $U_1 \cap U_2$ -isomorphism factors as a U_1 -isomorphism followed by a U_2 -isomorphism (and viceversa) so that we may choose t_2 to be a U_2 -isomorphism, up to possibly amplifying the fraction f_1 without changing it. Similarly, we can assume t_1 is a U_1 -isomorphism. By Lemma 2.8, the left “square” of (4) is weakly bicartesian. Therefore (weak push-out), g_1 and g_2 induce a morphism $f : a \rightarrow b$ such that $f \circ s_i = g_i$ for $i = 1, 2$. Hence $f = g_i s_i^{-1} = f_i$ on U_i as wanted.

Exactness at $\text{Mor}_{\mathcal{K}}(a, b)$: Let $f \in \text{Mor}_{\mathcal{K}}(a, b)$ be such that $\rho_i(f) = 0$ in $\mathcal{L}(U_i) = \mathcal{K}/\mathcal{K}_{Z_i}$ for $i = 1, 2$. This means that f factors through objects $c_i \in \mathcal{K}_{Z_i}$ as follows:

$$\begin{array}{ccccc} & & c_1 & & \\ & \alpha \swarrow & & \searrow f_1 & \\ a & & f & & b \\ & \beta \swarrow & & \searrow f_2 & \\ & & c_2 & & \end{array}$$

Take now x the weak push-out of α and β . By construction of the weak push-out (Def. 1.14), we have a distinguished triangle as in the first line of the diagram below. Since $(f_1 \ -f_2) \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = f - f = 0$, there exists a morphism $h : x \rightarrow b$ as follows:

$$\begin{array}{ccccccc} a & \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} & c_1 \oplus c_2 & \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} & x & \xrightarrow{s} & Ta \\ & & & \searrow \begin{pmatrix} f_1 & -f_2 \end{pmatrix} & \downarrow h & & \\ & & & & b & & \end{array}$$

We obtain a morphism $\frac{h}{s} = (Ta \xleftarrow{s} x \xrightarrow{h} b) \in \text{Mor}_{U_1 \cap U_2}(Ta, b)$. By Construction 2.9 and Remark 2.10, we have $\partial(\frac{h}{s}) = h \gamma \alpha = f_1 \alpha = f$.

Exactness at $\text{Mor}_{U_1 \cap U_2}(a, b)$: Let $\frac{f}{s} = (a \xleftarrow{s} x \xrightarrow{f} b)$ be a morphism over $U_1 \cap U_2$ such that $\partial(\frac{f}{s}) = 0$. As in Construction 2.9, choose a distinguished triangle

$$x \xrightarrow{s} a \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} c_1 \oplus c_2 \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} Tx$$

with $c_i \in \mathcal{K}_{Z_i}$. The assumption $\partial(\frac{f}{s}) = 0$ reads $Tf \gamma \alpha = 0$. Now apply Lemma 1.13 to the above triangle to produce objects $d, e \in \mathcal{K}$ and morphisms $\alpha_0, \beta_0, \gamma_0$ and δ_0 satisfying all the conclusions of Lemma 1.13, which the reader is encouraged to have at hand.

Claim : There exists a distinguished triangle of the form

$$(5) \quad b \xrightarrow{\gamma \alpha} Ta \xrightarrow{\begin{pmatrix} T\delta_0 \\ -T\gamma_0 \end{pmatrix}} Td \oplus Te \longrightarrow Tb.$$

Indeed, the composition $\alpha_2 \gamma_1 = T\delta_0 \gamma \gamma_1 = 0$ yields an Octahedron :

for some morphisms φ and ζ . Note in particular the distinguished triangle

$$b \xrightarrow{\gamma \alpha} Ta \xrightarrow{\begin{pmatrix} \varphi \\ -T\gamma_0 \end{pmatrix}} Td \oplus Te \xrightarrow{\zeta} Tb.$$

To obtain the triangle (5), observe that $(T\delta_0 - \varphi)\gamma = \alpha_2 - \alpha_2 = 0$. By the distinguished triangle over γ , there exists a morphism $h : Te \rightarrow Td$ such that $T\delta_0 - \varphi = h T\gamma_0$. Using this equality we get an isomorphism of triangles

$$\begin{array}{ccccccc} b & \xrightarrow{\gamma \alpha} & Ta & \xrightarrow{\begin{pmatrix} \varphi \\ -T\gamma_0 \end{pmatrix}} & Td \oplus Te & \xrightarrow{\zeta} & Tb \\ \parallel & & \parallel & & \downarrow \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} \simeq & & \parallel \\ b & \xrightarrow{\gamma \alpha} & Ta & \xrightarrow{\begin{pmatrix} T\delta_0 \\ -T\gamma_0 \end{pmatrix}} & Td \oplus Te & \xrightarrow{\zeta'} & Tb \end{array}$$

for $\zeta' := \zeta \cdot \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$. So, the lower triangle is distinguished. Hence the Claim.

Using this triangle and the assumption $Tf \circ \gamma \alpha = 0$ yields a factorisation of Tf as follows :

$$\begin{array}{ccccc} a & \xrightarrow{\gamma \alpha} & Tx & \xrightarrow{\begin{pmatrix} T\delta_0 \\ -T\gamma_0 \end{pmatrix}} & Td \oplus Te \longrightarrow Ta \\ & & \searrow Tf & & \downarrow \begin{pmatrix} Tf_1 & Tf_2 \end{pmatrix} \\ & & & & Tb \end{array}$$

for some morphisms $f_1 : d \rightarrow b$ and $f_2 : e \rightarrow b$. This reads $f = f_1 \delta_0 - f_2 \gamma_0$. Using the triangles of Lemma 1.13, it is easy to see that α_0, γ_0 are U_1 -isomorphisms and that β_0 and δ_0 are U_2 -isomorphisms. Consider now the morphisms $\frac{f_1}{\alpha_0} =$

$(a \xleftarrow{\alpha_0} d \xrightarrow{f_1} b)$ and $\frac{f_2}{\beta_0} = (a \xleftarrow{\beta_0} e \xrightarrow{f_2} b)$ in $\mathcal{L}(U_1)$ and $\mathcal{L}(U_2)$ respectively.

When restricted to $\mathcal{L}(U_1 \cap U_2)$ they clearly satisfy $\frac{f_1}{\alpha_0} - \frac{f_2}{\beta_0} = \frac{f_1 \delta_0}{\alpha_0 \delta_0} - \frac{f_2 \gamma_0}{\beta_0 \gamma_0} = \frac{f}{s}$. The last equation uses the relation $s = \alpha_0 \delta_0 = \beta_0 \gamma_0$ from Lemma 1.13. \square

3. GLUING OBJECTS

It is convenient to fix the following standard terminology.

Definition 3.1. Let $\mathrm{Spc}(\mathcal{K}) = U_1 \cup \dots \cup U_n$ be a covering by quasi-compact open subsets. Consider objects $a_i \in \mathcal{K}(U_i)$ and isomorphisms $\sigma_{ji} : a_i \xrightarrow{\sim} a_j$ on $U_i \cap U_j$ such that $\sigma_{ki} = \sigma_{kj} \sigma_{ji}$ on $U_i \cap U_j \cap U_k$ for $1 \leq i, j, k \leq n$. A *gluing of the objects a_i along the isomorphisms σ_{ij}* is an object $a \in \mathcal{K}$ and n isomorphisms $f_i : a \xrightarrow{\sim} a_i$ on U_i such that $\sigma_{ji} f_i = f_j$ on $U_i \cap U_j$ for all $1 \leq i, j \leq n$. An *isomorphism of gluings* $f : (a, f_1, \dots, f_n) \xrightarrow{\sim} (a', f'_1, \dots, f'_n)$ is an isomorphism $f : a \xrightarrow{\sim} a'$ in \mathcal{K} such that $f'_i f = f_i$ on U_i for all $i = 1, \dots, n$. (We temporarily dropped the mention of the restriction functors, for readability purposes.)

We first prove the gluing of objects without idempotent completions.

Lemma 3.2. *In a Mayer-Vietoris situation (Def. 2.1)*

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\rho_1} & \mathcal{L}(U_1) \\ \rho_2 \downarrow & & \downarrow \rho_{12} \\ \mathcal{L}(U_2) & \xrightarrow{\rho_{21}} & \mathcal{L}(U_1 \cap U_2), \end{array}$$

two objects $a_1 \in \mathcal{L}(U_1)$, $a_2 \in \mathcal{L}(U_2)$ with an isomorphism $\sigma : \rho_{12}(a_1) \xrightarrow{\sim} \rho_{21}(a_2)$ in $\mathcal{L}(U_1 \cap U_2)$ always admit a gluing (Def. 3.1).

Proof. The isomorphism σ can be represented by a fraction $a_1 \xleftarrow{s} x \xrightarrow{t} a_2$ where s, t both are $U_1 \cap U_2$ -isomorphisms. By Lemma 2.6 s and t factor as $s = s_1 s_2$ and $t = t_2 t_1$ where s_i, t_i are U_i -isomorphisms, see the upper part of Diagram (6). Now complete this diagram by taking the weak push-out of s_2 and t_1 :

$$(6) \quad \begin{array}{ccccc} a_1 & \xleftarrow{s} & x & \xrightarrow{t} & a_2 \\ & \swarrow s_1 & \downarrow s_2 & \downarrow t_1 & \searrow t_2 \\ & & y & & z \\ & & \downarrow u_1 & & \downarrow u_2 \\ & & & & a \end{array}$$

Applying Lemma 2.4, u_i is a U_i -isomorphism. The object a is then isomorphic to a_i over U_i via $f_1 := s_1 \circ u_1^{-1}$ and $f_2 := t_2 \circ u_2^{-1}$ respectively; the relation $\sigma f_1 = f_2$ is satisfied in $\mathcal{L}(U_1 \cap U_2)$ because of (6). \square

Theorem 3.3 (Gluing of two objects). *In a Mayer-Vietoris situation (Def. 2.1)*

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\rho_1} & \mathcal{K}(U_1) \\ \rho_2 \downarrow & & \downarrow \rho_{12} \\ \mathcal{K}(U_2) & \xrightarrow{\rho_{21}} & \mathcal{K}(U_1 \cap U_2), \end{array}$$

given two objects $a_i \in \mathcal{K}(U_i)$ for $i = 1, 2$ and an isomorphism $\sigma : \rho_{12}(a_1) \xrightarrow{\sim} \rho_{21}(a_2)$ in $\mathcal{K}(U_1 \cap U_2)$, there exists a gluing (Def. 3.1), which is unique up to (possibly non-unique) isomorphism.

Proof. Obviously $\sigma \oplus T\sigma : \rho_{12}(a_1 \oplus Ta_1) \simeq \rho_{21}(a_2 \oplus Ta_2)$. But $b_i := a_i \oplus Ta_i$ is an object of $\mathcal{L}(U_i)$ and since $\mathcal{L}(U_1 \cap U_2) \rightarrow \mathcal{L}(U_1 \cap U_2) = \mathcal{K}(U_1 \cap U_2)$ is fully faithful, the morphism $\sigma \oplus T\sigma$ is an isomorphism in $\mathcal{L}(U_1 \cap U_2)$ as well. By Lemma 3.2 there exists an object $b \in \mathcal{K}$ and isomorphisms $f_i : \rho_i(b) \rightarrow b_i$ in $\mathcal{L}(U_i)$ such that $(\sigma \oplus T\sigma) \circ \rho_{12}(f_1) = \rho_{21}(f_2)$. Consider now, for each $i = 1, 2$ the morphism $\pi_i : \rho_i(b) \rightarrow \rho_i(b)$ in $\mathcal{L}(U_i)$ defined by:

$$\rho_i(b) \xrightarrow{f_i} b_i \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} b_i \xrightarrow{f_i^{-1}} \rho_i(b),$$

π_i

where $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ on $b_i = a_i \oplus T(a_i)$ is the projection on a_i . Now, since the diagram

$$\begin{array}{ccccccc} \rho_{12}(b) & \xrightarrow{\rho_{12}(f_1)} & \rho_{12}(b_1) & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} & \rho_{12}(b_1) & \xrightarrow{\rho_{12}(f_1)^{-1}} & \rho_{12}(b) \\ \parallel & & \downarrow \sigma \oplus T\sigma & & \downarrow \sigma \oplus T\sigma & & \parallel \\ \rho_{21}(b) & \xrightarrow{\rho_{21}(f_2)} & \rho_{21}(b_2) & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} & \rho_{21}(b_2) & \xrightarrow{\rho_{21}(f_2)^{-1}} & \rho_{21}(b) \end{array}$$

is commutative, we have that $\rho_{12}(\pi_1) = \rho_{21}(\pi_2)$. We can now apply Mayer-Vietoris for morphisms (Thm. 2.11) to π_1 and π_2 to show that there exists $\pi : b \rightarrow b$ such that $\rho_i(\pi) = \pi_i$ for $i = 1, 2$. Consider now $h = \pi^2 - \pi$. Since $h = 0$ in $\mathcal{L}(U_i)$ it factors through objects $c_i \in \mathcal{K}_{Z_i}$ as follows:

$$\begin{array}{ccccc} b & \xrightarrow{h} & b & \xrightarrow{h} & b \\ & \searrow & \nearrow & \searrow & \nearrow \\ & c_1 & & c_2 & \end{array}$$

Since c_1 and c_2 have disjoint supports Proposition 1.6 shows that $h^2 = 0$. Then, by a standard trick, $p := \pi + h - 2\pi h$ satisfies $p^2 = p$ and still has the property $\rho_i(p) = \pi_i$ since $\rho_i(h) = 0$. Now, our category \mathcal{K} is idempotent complete by definition, so b splits as $b \simeq \text{Im}(p) \oplus \text{Ker}(p)$. Setting $a = \text{Im}(p)$ gives the desired object with the property $a \simeq a_i$ in $\mathcal{K}(U_i)$. Further details are left to the reader.

For uniqueness, suppose that (a, f_1, f_2) and (a', f'_1, f'_2) are two gluings. By Mayer-Vietoris for morphisms (Thm. 2.11) the morphisms $f_1^{-1} \circ f'_1$ and $f_2^{-1} \circ f'_2$ glue into a morphism $a' \rightarrow a$ which must be an isomorphism (Rem. 2.5). \square

Corollary 3.4 (Gluing three objects). *Let $\text{Spc}(\mathcal{K}) = U_1 \cup U_2 \cup U_3$ be a covering by quasi-compact open subsets. Consider three objects $a_i \in \mathcal{K}(U_i)$ for $i = 1, 2, 3$ and three isomorphisms $\sigma_{ij} : a_j \xrightarrow{\sim} a_i$ in $\mathcal{K}(U_i \cap U_j)$ for $1 \leq i < j \leq 3$ satisfying the cocycle relation $\sigma_{12} \circ \sigma_{23} = \sigma_{13}$ in $\mathcal{K}(U_1 \cap U_2 \cap U_3)$. Then they admit a gluing.*

Proof. Note that $\text{Spc}(\mathcal{K}(U_1 \cup U_2)) = U_1 \cup U_2 =: V$ by Proposition 1.11. Using gluing of two objects (Thm. 3.3), we can glue a_1 and a_2 into an object $b \in \mathcal{K}(V)$. Using Mayer-Vietoris for morphisms (Thm. 2.11) for the covering of $V \cap U_3$ by $U_1 \cap U_3$ and $U_2 \cap U_3$, we can construct a (non-unique) isomorphism between b and a_3 in $\mathcal{K}(V \cap U_3)$. By gluing of two objects (Thm. 3.3) for the covering of $\text{Spc}(\mathcal{K})$ given by V and U_3 , we can now glue b and a_3 into an object of \mathcal{K} . \square

Remark 3.5. As the above proof shows, the problem that arises with three open subsets is that the isomorphism between the objects $b \in \mathcal{K}(V)$ and $a_3 \in \mathcal{K}(U_3)$ on

$V \cap U_3$ is not unique. The various choices are parametrized by $\text{Mor}_{U_1 \cap U_2}(Ta_1, a_2)$ but we were not able to prove that two such choices yield isomorphic gluings and we tend to believe that this is wrong in general. Nevertheless, here is a case where the gluing works for several open subsets.

Theorem 3.6 (Connective gluing of several objects). *Let $\text{Spc}(\mathcal{K}) = U_1 \cup \dots \cup U_n$ be a covering by quasi-compact open subsets for $n \geq 2$. Consider objects $a_i \in \mathcal{K}(U_i)$ and isomorphisms $\sigma_{ji} : a_i \xrightarrow{\sim} a_j$ on $U_i \cap U_j$ satisfying the cocycle condition $\sigma_{kj}\sigma_{ji} = \sigma_{ki}$ for $1 \leq i, j, k \leq n$. Assume moreover the following Connectivity Condition: For any $i = 2, \dots, n$ and for any quasi-compact open $V \subset U_i$ (it is enough to take V a union of intersections of U_1, \dots, U_n), we suppose that:*

$$(7) \quad \text{Mor}_V(Ta_i, a_i) = 0.$$

Then there exists a gluing (Def. 3.1), which is unique up to unique isomorphism.

Proof. We prove the result by induction on n . Let us first establish the $n = 2$ case. By gluing of two objects (Thm. 3.3) we only need to prove the uniqueness of the isomorphism. To see this, it suffices to prove that for two gluings $a, a' \in \mathcal{K}$, two (iso)morphisms $g, g' : a \rightarrow a'$ which agree on U_1 and U_2 are equal. By Mayer-Vietoris for morphisms (Thm. 2.11), it suffices to show that $\text{Mor}_{U_1 \cap U_2}(Ta, a') = 0$ which follows from the Connectivity Condition (7) and from $a \simeq a' \simeq a_2$ on U_2 .

Suppose $n \geq 3$ and the result known for $n - 1$. Define $V = U_1 \cup \dots \cup U_{n-1}$. Since V is quasi-compact, we know by Proposition 1.11 that $\text{Spc}(\mathcal{K}(V)) = V$ and we can apply the induction hypothesis to construct a gluing $b \in \mathcal{K}(V)$ with isomorphisms $g_i : b \xrightarrow{\sim} a_i$ on U_i for $i = 1, \dots, n - 1$, such that $\sigma_{ij}g_j = g_i$ for all $1 \leq i, j \leq n - 1$. Consider the intersection $W := V \cap U_n$, which is covered by $n - 1$ quasi-compact subsets $W = (U_1 \cap U_n) \cup \dots \cup (U_{n-1} \cap U_n)$. In the category $\mathcal{K}(W)$, we have two objects b and a_n (i.e. their restrictions, of course) which are isomorphic on $U_i \cap U_n$ for $i = 1, \dots, n - 1$ in a compatible way with the σ_{ij} . By uniqueness of the gluing for $n - 1$, there exists a unique isomorphism $\sigma : b \xrightarrow{\sim} a_n$ on $V \cap U_n$ such that $\sigma_{in}\sigma = g_i$ for $i = 1, \dots, n - 1$. By the $n = 2$ case, we obtain the wanted gluing $a \in \mathcal{K}$ of b and a_n , unique up to unique isomorphism. Details are left to the careful reader. Note that the uniqueness of the isomorphism σ (at stage $n - 1$) is essential for the uniqueness of the gluing a (at stage n).

In the above induction, we needed that if the tuple $(U_1, \dots, U_n; a_1, \dots, a_n)$ satisfies the Connectivity Condition (7) for n , then:

- the tuple $(U_1, \dots, U_{n-1}; a_1, \dots, a_{n-1})$ satisfies (7) for $n - 1$,
- the tuple $(U_1 \cap U_n, \dots, U_{n-1} \cap U_n; a_1, \dots, a_{n-1})$ satisfies (7) for $n - 1$,
- the 4-uple $(U_1 \cup \dots \cup U_{n-1}, U_n; b, a_n)$ satisfies (7) for $n = 2$, for any object b .

These are easy to check. The last one comes from the assumption $i > 1$ in (7). \square

4. PICARD GROUPS

Definition 4.1. An object $a \in \mathcal{K}$ is called *invertible* if there exists an object b such that $a \otimes b \simeq 1$. By adjunction, see Def. 1.1, an object $a \in \mathcal{K}$ is invertible if and only if the evaluation map $\eta : Da \otimes a \rightarrow 1$ is an isomorphism.

Lemma 4.2. *An object a in \mathcal{K} is invertible if and only if it is invertible in \mathcal{K}/\mathcal{P} for all $\mathcal{P} \in \text{Spc}(\mathcal{K})$.*

Proof. This is clear by Proposition 1.7. \square

Definition 4.3. Define $\text{Pic}(\mathcal{K})$ to be the set of isomorphism classes of invertible objects in \mathcal{K} . The tensor product $\otimes : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ makes $\text{Pic}(\mathcal{K})$ into an abelian group with unit the class of 1.

Proposition 4.4. *Let X be a scheme and consider $D^{\text{perf}}(X)$ its derived category of perfect complexes. Then there is a split short exact sequence of abelian groups*

$$0 \rightarrow \text{Pic}(\mathcal{K}) \rightarrow \text{Pic}(D^{\text{perf}}(X)) \rightarrow C(X; \mathbb{Z}) \rightarrow 0$$

where $C(X; \mathbb{Z})$ stands for the group of locally constant functions from X to \mathbb{Z} .

Proof. We first describe $\text{Pic}(D^{\text{perf}}(X))$ where $X = \text{Spec}(R)$ is the spectrum of a local ring (R, \mathfrak{m}) . In this case, any object of $D^{\text{perf}}(R)$ is isomorphic to a so-called *minimal* complex of the form

$$C = \cdots \longrightarrow R^{\ell_i} \xrightarrow{d_i} R^{\ell_{i-1}} \longrightarrow \cdots$$

where, for all i , the differential d_i is a matrix with coefficients in \mathfrak{m} . If C is invertible in $D^{\text{perf}}(R)$ so is \bar{C} , its image under the functor $D^{\text{perf}}(R) \rightarrow D^{\text{perf}}(R/\mathfrak{m})$. But all the differentials of \bar{C} are 0 and the relation $C \otimes D \simeq R$, for some complex D , now shows that the complex C must be R concentrated in some degree, i.e. there exists $n_0 = n_0(C)$ such that $\ell_i = 1$ if $i = n_0$ and $\ell_i = 0$ otherwise.

For a global X , the map $\text{Pic}(D^{\text{perf}}(X)) \rightarrow C(X; \mathbb{Z})$ is now easily defined: for an invertible complex $C \in D^{\text{perf}}(X)$ and for $x \in X$ denote by C_x its image in $D^{\text{perf}}(\mathcal{O}_{X,x})$. The function $f_C : X \rightarrow \mathbb{Z}$ is then defined by $x \mapsto n_0(C_x)$. The rest of the proof is straightforward: a perfect complex which is locally trivial is quasi-isomorphic to its homology in degree zero and the latter must be a line bundle. \square

Definition 4.5. Define $\mathbb{G}_{\mathfrak{m}}(\mathcal{K}) = \text{Mor}_{\mathcal{K}}(1, 1)^{\times}$ to be the group of invertible elements of the (commutative) ring $\text{Mor}_{\mathcal{K}}(1, 1)$.

Theorem 4.6. *In a Mayer-Vietoris situation (Def. 2.1), there is an exact sequence of abelian groups*

$$\begin{aligned} \cdots \longrightarrow \text{Mor}_{\mathcal{K}(U_1 \cap U_2)}(T1, 1) \xrightarrow{1+\partial} \\ \xrightarrow{1+\partial} \mathbb{G}_{\mathfrak{m}}(\mathcal{K}) \longrightarrow \mathbb{G}_{\mathfrak{m}}(\mathcal{K}(U_1)) \oplus \mathbb{G}_{\mathfrak{m}}(\mathcal{K}(U_2)) \longrightarrow \mathbb{G}_{\mathfrak{m}}(\mathcal{K}(U_1 \cap U_2)) \xrightarrow{\delta} \\ \xrightarrow{\delta} \text{Pic}(\mathcal{K}) \longrightarrow \text{Pic}(\mathcal{K}(U_1)) \oplus \text{Pic}(\mathcal{K}(U_2)) \longrightarrow \text{Pic}(\mathcal{K}(U_1 \cap U_2)). \end{aligned}$$

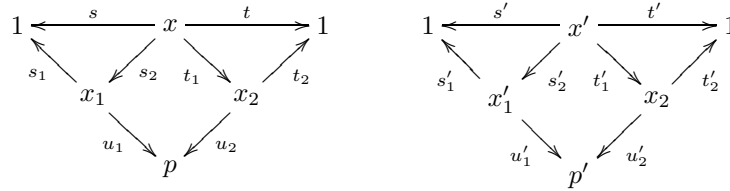
The homomorphism ∂ is as in Construction 2.9 and the unlabelled homomorphisms are the restrictions and the (multiplicative) differences thereof.

The homomorphism $\delta : \mathbb{G}_{\mathfrak{m}}(\mathcal{K}(U_1 \cap U_2)) \rightarrow \text{Pic}(\mathcal{K})$ is defined by gluing two copies of 1 by means of Theorem 3.3. Explicitly, it can be described as follows: Let $\sigma = ts^{-1} : 1 \xleftarrow{s} x \xrightarrow{t} 1$ in $\mathbb{G}_{\mathfrak{m}}(\mathcal{K}(U_1 \cap U_2))$, where s and t are $U_1 \cap U_2$ -isomorphisms; by Lemma 2.6, there exist factorizations $s = s_1 s_2$ and $t = t_2 t_1$ where s_i and t_i are U_i -isomorphisms; then $\delta(\sigma)$ is defined as the isomorphism class of the weak push-out $p \in \mathcal{K}$ of x_1 and x_2 over x

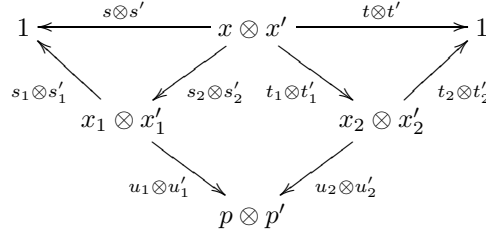
$$(8) \quad \begin{array}{ccccc} 1 & \xleftarrow{s} & x & \xrightarrow{t} & 1 \\ & \swarrow s_1 & \searrow s_2 & \swarrow t_1 & \searrow t_2 \\ & x_1 & & x_2 & \\ & \searrow u_1 & & \swarrow u_2 & \\ & & p & & \end{array}$$

Proof. First note that the homomorphism $1 + \partial : \text{Mor}_{U_1 \cap U_2}(T1, 1) \rightarrow \text{Mor}_{\mathcal{K}}(1, 1)$ lands in $\mathbb{G}_m(\mathcal{K})$. Indeed for any $g \in \text{Mor}_{U_1 \cap U_2}(T1, 1)$ one has $\partial(g) \circ \partial(g) = 0$, since $\partial(g) : 1 \rightarrow 1$ is zero in $\mathcal{K}(U_i)$ and hence factors via \mathcal{K}_{Z_i} for $i = 1, 2$ and since $\text{Mor}_{\mathcal{K}}(\mathcal{K}_{Z_1}, \mathcal{K}_{Z_2}) = 0$ by Prop. 1.6. So, $1 + \partial(g)$ is invertible with inverse $1 - \partial(g)$.

The connecting homomorphism $\delta : \mathbb{G}_m(\mathcal{K}(U_1 \cap U_2)) \rightarrow \text{Pic}(\mathcal{K})$ produces an object $p \in \mathcal{K}$, see Diagram (8), which is isomorphic to 1 on U_1 via $s_1 u_1^{-1}$ and on U_2 via $t_2 u_2^{-1}$, in a compatible way with σ on $U_1 \cap U_2$. The object p is then the gluing of two copies of 1 along the isomorphism σ on $U_1 \cap U_2$. Such a gluing is unique up to isomorphism by gluing of two objects (Thm. 3.3), and this gluing only depends on the map σ , and not on the various choices $(s, t, s_1, s_2, t_1, t_2, p)$. So, the map δ is well-defined and we now check that it is a group homomorphism. Take $1 \xleftarrow{s} x \xrightarrow{t} 1$ and $1 \xleftarrow{s'} x' \xrightarrow{t'} 1$ two units in $\mathbb{G}_m(\mathcal{K}(U_1 \cap U_2))$. With the same notations as above, factor these morphisms and perform the corresponding weak push-outs as follows



In a symmetric monoidal category, the composition of two morphisms between the unit object is also given by the tensor product. Hence we tensor together the two above diagrams to obtain the following one:



By Lemma 2.8, the above middle square is weakly bicartesian as well. Hence, $p \otimes p' = \delta(\sigma \otimes \sigma') = \delta(\sigma \sigma')$. This shows that δ is an abelian group homomorphism.

(Recall the restriction functors ρ_i and ρ_{ij} from Definition 2.1.) It is straightforward from the above definition of δ that $\rho_i \circ \delta = 1$ for $i = 1, 2$. To see that $\delta \circ \rho_{12} = 1$, for instance, one can assume that $s_2 = \text{id}$ and $t_2 = \text{id}_1$ in (8), in which case u_2 must also be an isomorphism, i.e. $p \simeq x_2 = 1$. The other compositions are clearly 1. The exactness of the left-hand side of the sequence up to $\mathbb{G}_m(\mathcal{K})$ follows from Mayer-Vietoris for morphisms (Thm. 2.11) applied at $a = b = 1$. It remains to check the exactness of the sequence at four spots.

Exactness at $\mathbb{G}_m(\mathcal{K}(U_1)) \oplus \mathbb{G}_m(\mathcal{K}(U_2))$: This is again immediate from Mayer-Vietoris for morphisms (Thm. 2.11) recalling that a local isomorphism is an isomorphism (Prop. 1.7).

Exactness at $\mathbb{G}_m(\mathcal{K}(U_1 \cap U_2))$: Let $\sigma = (1 \xleftarrow{s} x \xrightarrow{t} 1)$ in $\mathbb{G}_m(\mathcal{K}(U_1 \cap U_2))$ be such that $\delta(\sigma) \simeq 1$ in \mathcal{K} . This means that one can find a diagram of the form

$$\begin{array}{ccccc}
 1 & \xleftarrow{s} & x & \xrightarrow{t} & 1 \\
 & \swarrow s_1 & \searrow s_2 & \swarrow t_1 & \searrow t_2 \\
 & x_1 & & x_2 & \\
 & \searrow u_1 & & \swarrow u_2 & \\
 & & 1 & &
 \end{array}$$

see (8). One then sees two morphisms, namely $\sigma_1 = u_1 s_1^{-1} \in \mathbb{G}_m(\mathcal{K}(U_1))$ and $\sigma_2 = u_2 t_2^{-1} \in \mathbb{G}_m(\mathcal{K}(U_2))$, such that $\sigma_2^{-1} \circ \sigma_1 = \sigma$ in $\mathbb{G}_m(\mathcal{K}(U_1 \cap U_2))$.

Exactness at $\text{Pic}(\mathcal{K})$: Let p be an invertible object in \mathcal{K} such that $\rho_i(p) \simeq 1$ for $i = 1, 2$. Thus there exist U_i -isomorphisms $1 \xleftarrow{s_i} y_i \xrightarrow{t_i} p$. Performing the weak pull-back of s_1 and s_2 one obtains the diagram

$$\begin{array}{ccccc}
 1 & & y & & 1 \\
 & \swarrow t_1 & \searrow u_2 & \swarrow u_1 & \searrow t_2 \\
 & y_1 & & y_2 & \\
 & \searrow s_1 & & \swarrow s_2 & \\
 & & p & &
 \end{array}$$

and this defines $(1 \xleftarrow{t_1 u_2} y \xrightarrow{t_2 u_1} 1) \in \mathbb{G}_m(\mathcal{K}(U_1 \cap U_2))$. The image of this morphism under δ is clearly isomorphic to p by construction, see (8), the middle square in the above diagram being weakly bicartesian.

Exactness at $\text{Pic}(\mathcal{K}(U_1)) \oplus \text{Pic}(\mathcal{K}(U_2))$: This follows from gluing of two objects (Thm. 3.3) and from invertibility being a local property (see Lemma 4.2). \square

Theorem 4.7. *Let \mathcal{K} be an idempotent complete strongly closed tensor triangulated category. Suppose that $\text{Mor}_U(T1, 1) = 0$ for every quasi-compact open subsets $U \subset \text{Spc}(\mathcal{K})$. Then there exists a unique sheaf \mathbb{G}_m on $\text{Spc}(\mathcal{K})$, such that $\mathbb{G}_m(U) = \mathbb{G}_m(\mathcal{K}(U))$ when $U \subset \text{Spc}(\mathcal{K})$ is quasi-compact open. Moreover, there exists an injective homomorphism from the first Čech cohomology of $\text{Spc}(\mathcal{K})$ with coefficients in \mathbb{G}_m into the Picard group of \mathcal{K}*

$$\alpha : \check{H}^1(\text{Spc}(\mathcal{K}), \mathbb{G}_m) \hookrightarrow \text{Pic}(\mathcal{K})$$

which sends a \mathbb{G}_m -cocycle σ to the unique gluing of copies of 1 along the isomorphisms over the pairwise intersections given by σ , as described in Theorem 3.6.

Proof. We first prove by induction on n the following

Claim : Given a covering of a quasi-compact subset $V \subset \text{Spc}(\mathcal{K})$ by $n \geq 2$ quasi-compact open subsets, $V = U_1 \cup \dots \cup U_n$, and given morphisms $f_i : 1 \rightarrow 1$ over U_i such that $f_i = f_j$ over $U_i \cap U_j$ for $1 \leq i, j \leq n$, there exists a unique morphism $f : 1 \rightarrow 1$ over V such that $f = f_i$ over U_i .

Replacing \mathcal{K} by $\mathcal{K}(V)$, we can assume that $V = \text{Spc}(\mathcal{K})$. Now, for $n = 2$, this is Mayer-Vietoris for morphisms (Thm. 2.11). Note that unicity follows from $\text{Mor}_{U_1 \cap U_2}(T1, 1) = 0$. The induction on n is then easy: To construct f , glue the $n - 1$ first morphisms f_i into a morphism $g : 1 \rightarrow 1$ on $U_1 \cup \dots \cup U_{n-1}$ and show that it agrees with f_n on the intersection with U_n – this uses unicity for $n - 1$;

then apply the $n = 2$ case to glue g and f_n into a global f . To prove unicity of f , proceed similarly, using unicity for $n - 1$ and for $n = 2$ again. Hence the Claim.

Then the existence of the sheaf \mathbb{G}_m is immediate from the claim and from the fact that quasi-compact open subsets form a basis of the topology of $\mathrm{Spc}(\mathcal{K})$ ([1] Rem. 2.7 and Prop. 2.14). For the same reason and because of quasi-compactness of $\mathrm{Spc}(\mathcal{K})$, to define the homomorphism α , it suffices to consider \mathbb{G}_m -cocycles over finite coverings of $\mathrm{Spc}(\mathcal{K})$ by quasi-compact open subsets. In this situation, the gluing is guaranteed by Theorem 3.6. Hence α is well-defined.

Finally, injectivity of α is easy. Indeed, given a \mathbb{G}_m -cocycle σ over a covering $\mathrm{Spc}(\mathcal{K}) = U_1 \cup \dots \cup U_n$ with every U_i quasi-compact open, the gluing $a \in \mathrm{Pic}(\mathcal{K})$ comes with isomorphisms $f_i : a \xrightarrow{\sim} 1$ on each U_i , compatible with the $\sigma(U_i \cap U_j)$ as usual. Now, if $a = 1$, the latter compatibility means that the Čech boundary of the 0-cochain defined by the $f_i \in \mathbb{G}_m(U_i)$ is nothing but σ , that is, $\sigma = 0$ in $\check{H}^1(\mathrm{Spc}(\mathcal{K}), \mathbb{G}_m)$. \square

- Remarks 4.8.* (1) Note that the condition $\mathrm{Mor}_U(T1, 1) = 0$ does not hold in general, for instance in Modular Representation Theory, i.e. for $\mathcal{K} = kG$ -stab. For instance, for $k = \mathbb{F}_2$ and $G = \mathbb{Z}/2$, we even have $T1 \simeq 1!$
- (2) When the condition $\mathrm{Mor}_U(T1, 1) = 0$ holds for every quasi-compact open $U \subset \mathrm{Spc}(\mathcal{K})$ and when $\mathrm{Spc}(\mathcal{K})$ happens to be a scheme, Theorem 4.7 gives an injective homomorphism $\mathrm{Pic}(\mathrm{Spc}(\mathcal{K})) \hookrightarrow \mathrm{Pic}(\mathcal{K})$. In the case of $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ for X a scheme, this homomorphism is the one of Proposition 4.4.
- (3) The previous remark clearly leads us to glue copies of $T^r(1)$ for any $r \in \mathbb{Z}$, or even for any locally constant function $r \in C(\mathrm{Spc}(\mathcal{K}), \mathbb{Z})$. This induces a homomorphism:

$$\check{H}^1(\mathrm{Spc}(\mathcal{K}), \mathbb{G}_m) \times C(\mathrm{Spc}(\mathcal{K}), \mathbb{Z}) \longrightarrow \mathrm{Pic}(\mathcal{K})$$

which we do not know to be injective in general.

5. EXCISION

For later use, we state the following result in greater generality than in the Introduction. See Remark 2.2. The reader can as well consider the case of A and B reduced to a singleton, i.e. U open and Y closed.

Theorem 5.1 (Excision). *Let \mathcal{K} be an idempotent complete strongly closed tensor triangulated category and let $Y \subset U \subset \mathrm{Spc}(\mathcal{K})$. Assume that $Y = \cup_{\alpha \in A} Y_\alpha$ with every Y_α closed with quasi-compact complement and assume that $U = \cap_{\beta \in B} U_\beta$ with every U_β open and quasi-compact. Then the restriction functor $\rho : \mathcal{K} \rightarrow \mathcal{K}(U)$ induces an equivalence between the respective subcategories of objects supported on Y :*

$$\mathcal{K}_Y \xrightarrow{\sim} \mathcal{K}(U)_Y.$$

Proof. Remark first of all that $\mathrm{Spc}(\mathcal{K}(U)) \cong U$ by Prop. 1.11, whose proof generalizes *verbatim* to this situation.

Let us see that the functor $\rho : \mathcal{K}_Y \rightarrow \mathcal{K}(U)_Y$ is full. Given $a, b \in \mathcal{K}_Y$ and a fraction $a \xleftarrow{s} x \xrightarrow{f} b$ with s a U -isomorphism, we have $\mathrm{supp}(\mathrm{cone}(s)) \cap \mathrm{supp}(Ta) \subset \mathrm{supp}(\mathrm{cone}(s)) \cap U = \emptyset$, so $\mathrm{Mor}_{\mathcal{K}}(a, \mathrm{cone}(s)) = 0$ by Prop. 1.6. So, the morphism s is a split epimorphism, say $s \circ u = \mathrm{id}_a$ for a morphism $u : a \rightarrow x$. Amplifying the

fraction $f s^{-1}$ by u shows that this morphism $f s^{-1}$ is equal to (the restriction of) the morphism $f u : a \rightarrow b$.

Let us see that the functor $\rho : \mathcal{K}_Y \rightarrow \mathcal{K}(U)_Y$ is faithful. Let $f : a \rightarrow b$ be a morphism in \mathcal{K}_Y such that $\rho(f) = 0$, that is, there exists a U -isomorphism $s : x \rightarrow a$ such that $f s = 0$. As above, s must be a split epimorphism, hence $f = 0$.

Let us see that the functor $\rho : \mathcal{K}_Y \rightarrow \mathcal{K}(U)_Y$ is essentially surjective. Let $b \in \mathcal{K}(U)_Y$. There is an object $a \in \mathcal{K}$ such that $\rho(a) = b \oplus T(b)$. We have $\text{supp}(a) \cap U = \text{supp}_U(b) \cup \text{supp}_U(Tb) = \text{supp}_U(b) \subset Y$. So, if we call $Z = \text{Spc}(\mathcal{K}) \setminus U$ the complement of U , we have proved that $\text{supp}(a) \subset Y \cup Z$. Since $Y \cap Z = \emptyset$, we know by Theorem 1.8 that $a \simeq c \oplus d$ with $\text{supp}(c) \subset Y$ and $\text{supp}(d) \subset Z$. But then $\rho(a) = \rho(c)$ and we have found an object $c \in \mathcal{K}_Y$ such that $\rho(c) = b \oplus T(b)$. Now, consider the idempotent of $b \oplus T(b)$ corresponding to the projection on b . Since ρ is fully faithful, there exists a corresponding idempotent on the object c , which then decomposes accordingly, one factor going to b , as was to be shown. \square

Remark 5.2. If needed, the reader can formalize the following assertion: Given a point $\mathcal{P} \in \text{Spc}(\mathcal{K})$, the “local” category \mathcal{K}/\mathcal{P} , or rather its idempotent completion, is equivalent to the colimit of the categories $\mathcal{K}(U)$, over the quasi-compact open neighborhoods $U \ni \mathcal{P}$.

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